More Examples

Often curl$(\vec{F})$ is simpler than curl

Ex: Compute $\iint_S \text{Curl}(\vec{F})\cdot d\vec{S}$ for $\vec{F} = \langle xz, yz, xy\rangle$ and S the part of sphere $x^2+y^2+z^2=4$ inside cylider $x^2+y^2=1$ and above xy-plane.

Solution 1: Compute with Stokes

$$\text{curl}(\vec{F}) = \nabla\times\vec{F} = \det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\langle 1,1,0\rangle$$

$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), \sqrt{4-r^2}\rangle$ or $(r,\theta)\in[0,1]\times[0,2\pi]$

$$\vec{S}_r = \langle \cos(\theta), \sin(\theta), -\tfrac{1}{2}(4-r^2)^{-1/2}\cdot 2r\rangle = \langle\cos(\theta), \sin(\theta), -r(4-r^2)^{-1/2}\rangle$$

$$\vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$$

$$\vec{S}_r\times\vec{S}_\theta = \det\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(\theta) & \sin(\theta) & -r(4-r^2)^{-1/2} \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{vmatrix} = \langle r^2(4-r^2)^{-1/2}\cos\theta, r^2(4-r^2)^{-4}\sin\theta, r\rangle$$

$$\therefore \iint_S \text{Curl}(\vec{F})\,dS = \iint_D \text{Curl}(\vec{F})(\vec{S}(r,\theta))\cdot(S_r\times S_\theta)\,dA$$

$$= \iint_D (r\cos(\theta) - r\sin(\theta))\langle 1,1,0\rangle\cdot\langle r^2(4-r^2)^{-1/2}\cos\theta, r^2(4-r^2)^{-4}\sin\theta, r\rangle$$

$$= \iint_D r\cdot r^2(4-r^2)^{-1/2}(\cos^2(\theta)-\sin^2(\theta))\,dA$$

$$= \int_{\theta=0}^{3\pi}\cos 2\pi\int_0^1 r\cdot r^2(4-r^2)^{-1/2}\,dr\,d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi}\cos(2\theta)\int_{u=4}^{3}(4-u)u^{-1/2}\,du\,d\theta$$

$$= -\frac{1}{2}\int_{\theta=0}^{2\pi}\cos(\theta)\left[8u^{1/2}-\frac{2}{3}u^{3/2}\right]_4^3\,d\theta = \left(\frac{16}{3}-3\sqrt{3}\right)\cdot(0) = \boxed{0}$$

Sol 2: (with Stokes's Theorem).

$$\iint_S \text{Curl}(\vec{F})\,dS = \int_{\partial S}\vec{F}\cdot d\vec{r} \Rightarrow \text{Parametric } \partial S \text{ via}$$

$\vec{r}(\theta) = \langle \cos(\theta), \sin(\theta), \sqrt{3} \rangle$

$\vec{F}(\vec{r}(\theta)) = \langle \sqrt{3}\cos(\theta), \sqrt{3}\sin(\theta), \sin(\theta)\cos(\theta) \rangle$

$\vec{r}'(\theta) = \langle -\sin(\theta), \cos(\theta), 0 \rangle$

$\therefore \iint_S \mathrm{Curl}(\vec{F}) \cdot ds = \int_{\theta=0}^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot r'(\theta)\, d\theta = \int_{\theta=0}^{2\pi} 0 \, d\theta = \boxed{0}$

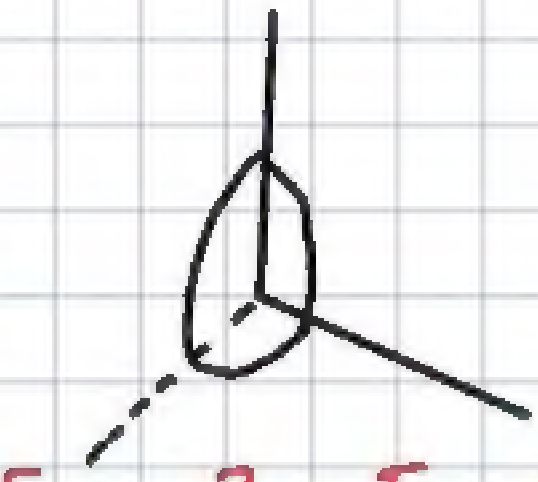
Note: ① often $\mathrm{Curl}(\vec{F})$ is simpler than $\vec{F}$

② The Stokes Equation also implies

$\iint_S \mathrm{curl}(\vec{F}) \cdot d\vec{s} = \iint_T \mathrm{Curl}(\vec{F}) \cdot ds \quad \Rightarrow (\partial S = \partial T)$

Ex: Compute $\int_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle xy, yz, zx \rangle$ and C the boundary of the part of $z = 1 - x^2 - y^2$ in the first octant.

Sol: Note that C has three pieces.



Solation: first, try Stokes's Theroem.

$\vec{S}(r,\theta) = \langle r\cos(\theta), r\sin(\theta), 1 - r^2 \rangle$ or $(r,\theta) \in [0,1] \times [0, \frac{\pi}{2}]$

$$\mathrm{Curl}(\vec{F}) = \nabla \times \vec{F} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \langle y, z, x \rangle$$

$\therefore \mathrm{Curl}(\vec{F})(\vec{S}(r,\theta)) = -\langle \sin(\theta), 1 - r^2, \cos(\theta) \rangle$

$\vec{S}_r = \langle \cos\theta, \sin(\theta), -2r \rangle \quad \vec{S}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$

$$\vec{S}_r \times \vec{S}_\theta = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos(\theta) & 0 \end{vmatrix} = \langle 2r^2\cos\theta, (2r^2\sin(\theta)), r\cos^2\theta + r\sin^2\theta \rangle$$

Sol: $\therefore \mathrm{Curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)$

$= -r(2r^2\sin(\theta)\cos(\theta) + 2(1-r^2)r\sin(\theta) + r\cos(\theta))$

$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \mathrm{Curl}(\vec{F}) \cdot d\vec{S}$

$= \iint_D \mathrm{Curl}(\vec{F})(\vec{S}(r,\theta)) \cdot (\vec{S}_r \times \vec{S}_\theta)\, dA$

$= \int_0^1 \int_0^{\pi/2} -r^2(r\sin(2\theta) + 2(1-r^2)\sin(\theta) + \cos(\theta))\, d\theta$

$$= \int_0^1 -r^2(r-2(1-r^2)+1)\,dr = \int_0^1 -r^2(2r^2+r-1)\,dr$$

$$= \int_0^1 -2r^4 - r^3 + r^2\,dr$$

$$= \boxed{\frac{2}{5} - \frac{1}{4} - 1}$$